obtaining $10x = 73.1414\cdots$. We now multiply by a power of 10 to move one block to the left of the decimal point; here getting $1000x = 7314.1414\cdots$. We now subtract to obtain an integer; here getting $1000x - 10x = 7314 - 73 = 7241$, whence $x = 7241/990$, a rational number.

Cantor's Second Proof

We will now give Cantor's second proof of the uncountability of $\mathbb{R}$. This is the elegant "diagonal" argument based on decimal representations of real numbers.

2.5.5 Theorem *The unit interval* $[0, 1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$ *is not countable.*

Proof. The proof is by contradiction. We will use the fact that every real number $x \in [0, 1]$ has a decimal representation $x = 0.b_1b_2b_3\cdots$, where $b_i = 0, 1, \ldots, 9$. Suppose that there is an enumeration $x_1, x_2, x_3\cdots$ of all numbers in [0,1], which we display as:

$$\begin{aligned} x_1 &= 0.b_{11}b_{12}b_{13}\cdots b_{1n}\cdots, \\ x_2 &= 0.b_{21}b_{22}b_{23}\cdots b_{2n}\cdots, \\ x_3 &= 0.b_{31}b_{32}b_{33}\cdots b_{3n}\cdots, \\ \cdots & \qquad \cdots \\ x_n &= 0.b_{n1}b_{n2}b_{n3}\cdots b_{nn}\cdots, \\ \cdots & \qquad \cdots \end{aligned}$$

We now define a real number $y := 0.y_1y_2y_3\cdots y_n\cdots$ by setting $y_1 := 2$ if $b_{11} \ge 5$ and $y_1 := 7$ if $b_{11} \le 4$; in general, we let

$$y_n := \begin{cases} 2 & \text{if } b_{nn} \ge 5, \\ 7 & \text{if } b_{nn} \le 4. \end{cases}$$

Then $y \in [0, 1]$. Note that the number y is not equal to any of the numbers with two decimal representations, since $y_n \ne 0, 9$ for all $n \in \mathbb{N}$. Further, since y and x_n differ in the nth decimal place, then $y \ne x_n$ for any $n \in \mathbb{N}$. Therefore, y is not included in the enumeration of [0,1], contradicting the hypothesis. Q.E.D.

Exercises for Section 2.5

1. If $I := [a, b]$ and $I' := [a', b']$ are closed intervals in $\mathbb{R}$, show that $I \subseteq I'$ if and only if $a' \le a$ and $b \le b'$.
2. If $S \subseteq \mathbb{R}$ is nonempty, show that S is bounded if and only if there exists a closed bounded interval I such that $S \subseteq I$.
3. If $S \subseteq \mathbb{R}$ is a nonempty bounded set, and $I_S := [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval containing S, show that $I_S \subseteq J$.
4. In the proof of Case (ii) of Theorem 2.5.1, explain why x, y exist in S.
5. Write out the details of the proof of Case (iv) in Theorem 2.5.1.
6. If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a nested sequence of intervals and if $I_n = [a_n, b_n]$, show that $a_1 \le a_2 \le \cdots \le a_n \le \cdots$ and $b_1 \ge b_2 \ge \cdots \ge b_n \ge \cdots$.
7. Let $I_n := [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.
8. Let $J_n := (0, 1/n)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} J_n = \emptyset$.
9. Let $K_n := (n, \infty)$ for $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

10. With the notation in the proofs of Theorems 2.5.2 and 2.5.3, show that we have $\eta \in \bigcap_{n=1}^{\infty} I_n$. Also show that $[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$.
11. Show that the intervals obtained from the inequalities in (2) form a nested sequence.
12. Give the two binary representations of $\frac{3}{8}$ and $\frac{7}{16}$.
13. (a) Give the first four digits in the binary representation of $\frac{1}{3}$.
    (b) Give the complete binary representation of $\frac{1}{3}$.
14. Show that if $a_k, b_k \in \{0, 1, \ldots, 9\}$ and if

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_m}{10^m} \neq 0,$$

then $n = m$ and $a_k = b_k$ for $k = 1, \ldots, n$.

15. Find the decimal representation of $-\frac{2}{7}$.
16. Express $\frac{1}{7}$ and $\frac{2}{19}$ as periodic decimals.
17. What rationals are represented by the periodic decimals $1.25137 \cdots 137 \cdots$ and $35.14653 \cdots 653 \cdots$?

If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n \leq (c-1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \leq (c-1)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n},$$

from which it follows that $0 < h_n < 1/nc$ for $n \in \mathbb{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc}$$

so that

$$|c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + \tfrac{1}{2}n(n-1)k_n^2 + \cdots \geq 1 + \tfrac{1}{2}n(n-1)k_n^2,$$

whence it follows that

$$n - 1 \geq \tfrac{1}{2}n(n-1)k_n^2.$$

Hence $k_n^2 \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$. □

Exercises for Section 3.1

1. The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case:

   (a) $x_n := 1 + (-1)^n$,

   (b) $x_n := (-1)^n/n$,

   (c) $x_n := \dfrac{1}{n(n+1)}$,

   (d) $x := \dfrac{1}{n^2 + 2}$.

2. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) 5, 7, 9, 11, . . . ,
   (b) 1/2, –1/4, 1/8, –1/16, . . .,
   (c) 1/2, 2/3, 3/4, 4/5, . . . ,
   (d) 1, 4, 9, 16, ...
3. List the first five terms of the following inductively defined sequences.
   (a) $x_1 := 1, \quad x_{n+1} := 3x_n + 1,$
   (b) $y_1 := 2, \quad y_{n+1} := \frac{1}{2}(y_n + 2/y_n),$
   (c) $z_1 := 1, \quad z_2 := 2, \ z_{n+2} := (z_{n+1} + z_n)/(z_{n+1} - z_n),$
   (d) $s_1 := 3, \quad s_2 := 5, \ s_{n+2} := s_n + s_{n+1}.$
4. For any $b \in \mathbb{R}$, prove that $\lim(b/n) = 0$.
5. Use the definition of the limit of a sequence to establish the following limits.
   (a) $\lim\left(\frac{n}{n^2+1}\right) = 0,$
   (b) $\lim\left(\frac{2n}{n+1}\right) = 2,$
   (c) $\lim\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2},$
   (d) $\lim\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}.$
6. Show that
   (a) $\lim\left(\frac{1}{\sqrt{n+7}}\right) = 0,$
   (b) $\lim\left(\frac{2n}{n+2}\right) = 2,$
   (c) $\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0,$
   (d) $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0.$
7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.
   (a) Use the definition of limit to show that $\lim(x_n) = 0$.
   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.
8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).
9. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.
10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.
11. Show that $\lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0.$
12. Show that $\lim(\sqrt{n^2+1} - n) = 0$.
13. Show that $\lim(1/3^n) = 0$.
14. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$. [*Hint*: Use the Binomial Theorem as in Example 3.1.11(d).]
15. Show that $\lim\left((2n)^{1/n}\right) = 1$.
16. Show that $\lim(n^2/n!) = 0$.
17. Show that $\lim(2^n/n!) = 0$. [*Hint*: If $n \geq 3$, then $0 < 2^n/n! \leq 2\left(\frac{2}{3}\right)^{n-2}$.]
18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.